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THE PARACONSISTENT LOGIC Z
A possible solution to Jaśkowski's problem*

Abstract. We present a paraconsistent logic, called **Z**, based on an intuitive possible worlds semantics, in which the replacement theorem holds. We show how to axiomatize this logic and prove the completeness theorem.

1. Jaśkowski's problem and paraconsistent logic

It seems that nowadays the main open problem in the field of paraconsistent logic is still : Does paraconsistent logic really exist?

This means: can we find a good paraconsistent logic? Of course one logic may be good for Mr. Black and bad for Mr. White. In other words, there is no formal definition of what is a good logic. Anyway all the known systems presented thus far bear serious defects. At least there is no paraconsistent logic which is recognized unanimously as a good paraconsistent logic.

The central problem of paraconsistent logic is to find a negation which is a *paraconsistent* negation in the sense that $a, \neg a \nvdash b$, and which at the same time is a paraconsistent *negation* in the sense that it has enough strong properties to be called a negation (cf. Béziau 00). Furthemore, such kind of paraconsistent negation, if it has not to be an artificial construct, a mere abstract object of a formal meaningless game, must have an intuitive background.

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Jaśkowski in his 1948's paper (Jaśkowski 48) already clearly stated the problem of combining these three features, and this has been called Jaśkowski's problem, cf. e.g. Kotas/da Costa 77. It is not misleading to say that Jaśkowski's problem is the basic problem of paraconsistent logic.

Jaśkowski himself presented a system, called discussive logic, but it does not seem to be a solution to his problem; for recent survey and discussion about Jaśkowski's logic see (da Costa/Doria 95) and (Urchs 95). Later on further systems were presented, see e.g. da Costa 63, D'Ottaviano/da Costa 70, Priest 79, which don't either appear as solutions.

Here we present a paraconsistent logic Z which seems to be a possible solution to Jaśkowski's problem: this logic is paraconsistent, has a very intuitive semantics, is axiomatizable and its negation is quite strong, in particular the replacement theorem holds for it. Funny enough the logic Z is closely connected to the modal logic **S5** which is the basis of Jaśkowski's discussive logic.

2. Possible worlds semantics for Z

2.1. Intuitive explanation

Semantically speaking, the basic idea of paraconsistent negation is that a proposition and its negation can both be true. In modern logic, even from the viewpoint of classical logic, true does not mean necessary true in the real world, but true in some possible worlds, or some models. Note therefore that the idea of paraconsistent negation is not so strange and does not commit one to believe in true contradictions.

The point is that for paraconsistent logic we have to consider paraconsistent worlds (or structures, or models), i.e. worlds in which a proposition and its negation can both be true.

The basic idea of possible worlds semantics is to semantically define connectives using packages of possible worlds with (Kripke) or without (Carnap) a relation of accessibility between them. Let us call here such a package simply a *cosmos*. Validity and semantical consequences are defined by considering all the possible cosmoses of possible worlds.

In a given cosmos, we will say that $\neg a$, the paraconsistent negation of the proposition a, is false in a possible world iff a is true in all possible worlds of the cosmos. If we have a cosmos where there is a possible world W in which a is true and a possible world V in which a is false, therefore there exists a world in which $\neg a$ is true, this can be the world W itself. In this case W is a paraconsistent world.

We don't consider here relations of accessibility. This is the same as to consider a universal accessibility relation (every world is accessible from every world), as it is known from the case of **S5**. The logic **Z** has in fact a close connection with **S5**: $\neg a$ in **Z** means $\sim\Box a$ in **S5**, where $\sim$ is the classical negation. The logic **Z** is translatable in **S5** and **S5** contains **Z** in the sense that **Z** is a reduct of **S5** (in the sense of model theory).

It is of course possible to consider any other modal logic and to extract a paraconsistent logic from it, defining in the same way the paraconsistent negation as $\sim\Box$.

Another possibility would be to say that $\neg a$ is false in a possible world of a given cosmos iff a is true in *most of* the worlds of this cosmos, using a definition of *most of* given for example in (Carnielli/Veloso 97).

What about the relation between the semantics of **Z** and Kripke semantics for intuitionistic logic? The condition for negation is dual of the condition of **Z**, but in the intuitionistic case a specific accessibility relation is needed: $\neg a$ is true in a given world iff a is false in all possible accessible worlds. Moreover in the intuitionistic case, the accessibility relation is also used in order to define implication. This is not the case of **Z**, where the implication is classical.

But it is clear that the possible worlds semantics dual of Kripke semantics for intuitionistic logic leads to a paraconsistent logic. One question is to know if this paraconsistent logic is similar to the paraconsistent logics dual of intuitionistic logic from an algebraic viewpoint (Sette/Alves/Queiroz 95) or sequent calculus viewpoint (Urbas 96).

2.2. Mathematical definition

We consider a standard set of zero-order formulas $\mathsf{For}_{\mathbf{Z}}$ built with three binary connectives $\wedge$, $\vee$, $\rightarrow$, and one unary connective $\neg$.

To simplify the definition we consider here bivaluations, i.e., functions from $\mathsf{For}_{\mathbf{Z}}$ to $\{0,1\}$, rather than possible worlds. Therefore a cosmos is here a set of bivaluations.

DEFINITION 2.1. A **Z**-*cosmos* is any non-empty set $\mathbb{C}$ of bivaluations defined by: $\alpha \in \mathbb{C}$ iff it obeys the classical conditions for conjunction ($\wedge$), disjunction ($\vee$), implication ($\rightarrow$) and moreover obeys the following condition for the connective $\neg$ intended to be a paraconsistent negation:

$$(\neg\text{f}) \qquad \alpha(\neg a) = 0 \quad \text{iff} \quad \forall_{\beta \in \mathbb{C}}\, \beta(a) = 1.$$

Remark 1. If we replace the condition (¬f) by the following condition (¬t), then we get the same notion of **Z**-cosmos:

(¬t) $\alpha(\neg a) = 1 \quad \text{iff} \quad \exists_{\beta \in \mathbb{C}}\ \beta(a) = 0.$

Remark 2. Given any bivaluation α of any **Z**-cosmos we have:

(¬r) If $\alpha(a) = 0$ then $\alpha(\neg a) = 1$,

(¬rr) If $\alpha(\neg a) = 0$ then $\alpha(a) = 1$,

(¬ft) If $\alpha(\neg a) = 0$ then $\forall_{\beta \in \mathbb{C}}\ \beta(\neg a) = 0$.

DEFINITION 2.2. A formula a is *valid in a cosmos* $\mathbb{C}$ iff its value is one for all bivaluations of $\mathbb{C}$, i.e., $\forall_{\alpha \in \mathbb{C}}\ \alpha(a) = 1$.

DEFINITION 2.3. A formula a is **Z**-*valid* (notation $\vDash_{\mathbf{Z}} a$) iff it is valid in all **Z**-cosmoses.

DEFINITION 2.4. A formula a is a *consequence* of a theory T in a **Z**-cosmos $\mathbb{C}$ iff for any bivaluation α of $\mathbb{C}$ if $\alpha(b) = 1$, for all formulas b of T, then $\alpha(a) = 1$.

DEFINITION 2.5. A formula a is a **Z**-*consequence* of a theory T (notation $T \vDash_{\mathbf{Z}} a$) iff a is a consequence of T in all **Z**-cosmoses.

DEFINITION 2.6. We consider the following *translation function* * from the set of formulas of **Z** into the set of formulas of **S5**; $\langle \mathsf{For}_{\mathbf{S5}}; \wedge, \vee, \to, \sim, \Box \rangle$ ($\sim$ is the classical negation).

$$\begin{aligned} a^* &= a && \text{if } a \text{ is atomic}, \\ (a \circ b)^* &= a^* \circ b^* && \circ \in \{\wedge, \vee, \to\}, \\ (\neg a)^* &= \sim \Box a^* . \end{aligned}$$

THEOREM 2.1. $T \vDash_{\mathbf{Z}} a \quad \textit{iff} \quad T^* \vDash_{\mathbf{S5}} a^*$

COROLLARY 2.1. *The logic* **Z** *is strictly paraconsistent. That is to say given any schema b which is not tautological (cf. Urbas 1990), we have:*

$$a, \neg a \nvDash_{\mathbf{Z}} b$$

FACT 2.1. (i) *All substitutions of theses of* $\to$-$\wedge$-$\vee$-*fragment of classical sentential calculus are* **Z**-*valid.*

(ii) *The following formulas are* **Z**-*valid:*

(1) $$a \vee \neg a$$

(2) $$\neg(a \wedge \neg a)$$

(3) $$(a \to \neg a) \to \neg a$$

(4) $$(\neg a \to a) \to a$$

(5) $$\neg\neg a \to a$$

(6) $$(a \to b) \to \neg a \vee b$$

(7) $$\neg(a \vee b) \to \neg a \wedge \neg b$$

(8) $$\neg(a \wedge b) \to \neg a \vee \neg b$$

(9) $$\neg(\neg a \vee \neg b) \to a \wedge b$$

(10) $$\neg(\neg a \wedge \neg b) \to a \vee b$$

(11) $$\neg(a \vee \neg b) \to \neg a \wedge b$$

(12) $$\neg(a \wedge \neg b) \to \neg a \vee b$$

(13) $$\neg(\neg a \vee b) \to a \wedge \neg b$$

(14) $$\neg(\neg a \wedge b) \to a \vee \neg b$$

(15) $$(a \wedge \neg b) \wedge \neg(a \wedge \neg b) \to (a \wedge \neg a)$$

(16) $$\neg(a_1 \wedge \cdots \wedge a_n) \to (\neg a_1 \vee \cdots \vee \neg a_n)$$

PROOF. (i) From definitions 2.1–2.3.

(ii) We prove directly only sample examples. The reader expert in modal logic can check indirectly the results *via* the translation of **Z** into **S5**.

(1): Suppose that we have a bivaluation α in a **Z**-cosmos $\mathbb{C}$ such that $\alpha(\neg(a \wedge \neg a)) = 0$. Then by $(\neg\text{f})$, for all $\beta \in \mathbb{C}$: $\beta(a \wedge \neg a) = 1$, i.e., $\beta(a) = 1$ and $\beta(\neg a) = 1$. But if $\beta(\neg a) = 1$ by $(\neg\text{t})$, there exists $\gamma \in \mathbb{C}$ such that $\gamma(a) = 0$, which is absurd.

(4): Suppose that we have a bivaluation α in a **Z**-cosmos $\mathbb{C}$ such that $\alpha(a) = 0$. By $(\neg\text{rr})$, $\alpha(\neg a) = 1$, therefore $\alpha(\neg a \to a) = 0$.

(5): Suppose that we have a bivaluation α in a **Z**-cosmos $\mathbb{C}$ such that $\alpha(\neg\neg a) = 1$, then by $(\neg\text{t})$, there exists $\beta \in \mathbb{C}$ such that $\beta(\neg a) = 0$. By $(\neg\text{f})$, $\alpha(a) = 1$.

(6): Suppose that we have a bivaluation α in a Z-cosmos $\mathbb{C}$ such that $\alpha(\neg a \vee b) = 0$, then $\alpha(\neg a) = 0$ and $\alpha(b) = 0$. By $(\neg\text{rr})$, $\alpha(a) = 1$, therefore $\alpha(a \to b) = 0$.

(15): Suppose that $\alpha(a \wedge \neg b) = 1 = \alpha(\neg(a \wedge \neg b))$. If $\alpha(a \wedge \neg b) = 1$, then $\alpha(a) = 1$ and $\alpha(\neg b) = 1$. Suppose that $\alpha(\neg a) = 0$, i.e., by $(\neg\text{f})$, $\forall_{\beta \in \mathbb{C}}\ \beta(a) = 1$.

If $\alpha(\neg(a \wedge b)) = 1$, then there exists $\gamma \in \mathbb{C}$ such that $\gamma(a \wedge \neg b) = 0$, i.e., $\gamma(a) = 0$ or $\gamma(\neg b) = 0$. The only possibility is $\gamma(\neg b) = 0$. So, by (¬ft), $\forall_{\beta \in \mathbb{C}}\ \beta(\neg b) = 0$, and in particular, $\alpha(\neg b) = 0$, which is absurd.

(16): Suppose there is a a bivaluation α of a **Z**-cosmos $\mathbb{C}$ such that $\alpha(\neg(a_1 \wedge \cdots \wedge a_n)) = 1$ and $\alpha(\neg a_1 \vee \cdots \vee \neg a_n) = 0$. By (¬t), there exists $\beta \in \mathbb{C}$ such that $\beta(a_1 \wedge \cdots \wedge a_n) = 0$, i.e., $\beta(a_j) = 0$ for some j $(1 \leq j \leq n)$. On the other hand $\alpha(\neg a_i) = 0$ for every i $(1 \leq i \leq n)$, in particular $\alpha(\neg a_j) = 0$. Hence by (¬f), for all $\gamma \in \mathbb{C}$: $\gamma(a_j) = 1$, which is absurd. □

Fact 2.2. (i) *If* $\vDash_{\mathbf{Z}} a \to b$ *and* $\vDash_{\mathbf{Z}} a$, *then* $\vDash_{\mathbf{Z}} b$.

(ii) *If* $\vDash_{\mathbf{Z}} a \to b$ *then* $\vDash_{\mathbf{Z}} \neg(a \wedge \neg b)$.

Proof. (i) From definitions 2.1–2.3.

(ii) Suppose that $\nvDash_{\mathbf{Z}} \neg(a \wedge \neg b)$. Therefore there are a **Z**-cosmos $\mathbb{C}$ and $\beta \in \mathbb{C}$ such that $\beta(\neg(a \wedge \neg b) = 0$. Then by (¬f), for all $\gamma \in \mathbb{C}$: $\gamma(a \wedge \neg b) = 1$, i.e., $\gamma(a) = 1$ and $\gamma(\neg b) = 1$. So, by (¬t), there is $\alpha \in \mathbb{C}$ such that $\alpha(b) = 0$. Because for all $\gamma \in \mathbb{C}$, $\gamma(a) = 1$, then $\alpha(a) = 1$. Therefore we have $\alpha(a \to b) = 0$. So $\nvDash_{\mathbf{Z}} a \to b$. □

Lemma 2.1. *If the values of two formulas are the same for every* bivaluations *of a given* **Z***-cosmos, then the values of their negations are the same for every bivaluations of this* **Z***-cosmos.*

Proof. Suppose for all $\alpha \in \mathbb{C}$, $\alpha(a) = \alpha(b)$ and there exists $\alpha' \in \mathbb{C}$ such that $\alpha'(\neg a) \neq \alpha'(\neg b)$. Suppose $\alpha'(\neg a) = 1$ and $\alpha'(\neg b) = 0$. Then, by (¬t), there exists $\beta \in \mathbb{C}$ such that $\beta(a) = 0$ and, by (¬f), for all $\gamma \in \mathbb{C}$, $\gamma(b) = 1$, in particular $\beta(b) = 1$, this contradicts the hypothesis. □

Corollary 2.2. *The bi-implication* $\leftrightarrow$ *standardly defined with* $\to$ *and* $\wedge$ *is a congruence relation for* **Z**, *i.e., the replacement theorem holds for* $\leftrightarrow$ *in* **Z**.

3. Axiomatization of Z

3.1. The Hilbertian system HZ

Definition 3.1. The *system* **HZ** contains the usual axioms for positive classical logic (→-∧-∨-fragment of classical sentential calculus), i.e., for all $a, b, c \in \mathsf{For}_{\mathbf{Z}}$:

(AP1) $$a \to (b \to a)$$

(AP2) $$(a \to (b \to c)) \to ((a \to b) \to (a \to c))$$

(AP3) $$((a \to b) \to a) \to a$$

(AP4) $$a \wedge b \to a$$

(AP5) $$a \wedge b \to b$$

(AP6) $$a \to (b \to (a \wedge b))$$

(AP7) $$a \to a \vee b$$

(AP8) $$b \to a \vee b$$

(AP9) $$(a \to c) \to ((b \to c) \to (a \vee b \to c))$$

and the following axioms for all $a, b \in \mathsf{For}_{\mathbf{Z}}$:

(AZ1) $$a \vee \neg a$$

(AZ2) $$(a \wedge \neg b) \wedge \neg(a \wedge \neg b) \to (a \wedge \neg a)$$

(AZ3) $$\neg(a \wedge b) \to (\neg a \vee \neg b)$$

(AZ4) $$\neg\neg a \to a$$

The system **HZ** contains the rules:

(MP) $$\frac{a \to b \qquad a}{b}$$

(RZ) $$\frac{a \to b}{\neg(a \wedge \neg b)}$$

DEFINITION 3.2. A formula a is *provable* in **HZ** (notation $\vdash_{\mathbf{Z}} a$) iff there exists a finite sequence of formulas $b_1, \ldots, b_n$ such that $b_n = a$ and b_i $(1 \leq i \leq n)$ is an axiom of **HZ** or is the conclusion of a rule whose premises are among $b_1, \ldots, b_m$ $(m < i)$.

LEMMA 3.1. *For any $n \in \omega$ the following formula is provable in* **HZ***:*

($\mathrm{AZ3}_n$) $$\neg(a_1 \wedge \cdots \wedge a_n) \to (\neg a_1 \vee \cdots \vee \neg a_n)$$

PROOF. Induction on the number n. As inductive hypotesis, let us assume that the following formula is provable in **HZ**:

$(*)$ $$\neg(a_1 \wedge \cdots \wedge a_{n-1}) \to (\neg a_1 \vee \cdots \vee \neg a_{n-1})$$

By (AZ3) we have that the following thesis of **HZ**:

$(**)$ $$\neg((a_1 \wedge \cdots \wedge a_{n-1}) \wedge a_n) \to (\neg(a_1 \wedge \cdots \wedge a_{n-1}) \vee \neg a_n)$$

For all $a, b, c, d \in \mathsf{For}_{\mathbf{Z}}$ the following formula is provable from (AP1)–(AP9):

$$(***) \qquad (a \to b \vee c) \to ((b \to d) \to (a \to d \vee c))$$

We put: $a = \neg((a_1 \wedge \cdots \wedge a_{n-1}) \wedge a_n)$, $b = \neg(a_1 \wedge \cdots \wedge a_{n-1})$, $c = \neg a_n$ and $d = (\neg(a_1 \wedge \cdots \wedge a_{n-1})$. We have (AZ3$_n$), by (MP) and $(*)$–$(***)$. □

DEFINITION 3.3. A formula a is *deducible* in **HZ** from a theory T (notation $T \vdash_{\mathbf{Z}} a$) iff there are $n \geq 0$, $b_1, \ldots, b_n \in T$ such that $\vdash_{\mathbf{Z}} b_1 \wedge \cdots \wedge b_n \to a$.[2]

3.2. Soundness and completeness

THEOREM 3.1 (Soundness). *If $T \vdash_{\mathbf{Z}} a$ then $T \vDash_{\mathbf{Z}} a$.*

PROOF. By facts 2.1 and 2.2. □

DEFINITION 3.4. A *maximal* theory M of **HZ** is a theory which is non-trivial (i.e. there exists a such that $M \nvdash_{\mathbf{Z}} a$) and has no non-trivial proper extensions.

LEMMA 3.2 (Lindenbaum Lemma). *If $T \nvdash_{\mathbf{Z}} a$, there exists a maximal theory M which is an extension of T, such that $M \nvdash_{\mathbf{Z}} a$.*

PROOF. As it is known (Lindenbaum-Asser Theorem), for any relation $\vdash_{\mathbf{Z}}$-verifying monotonicity and finiteness, the following results holds: if $T \nvdash_{\mathbf{Z}} a$, then there exists an extension R of T such that $R \nvdash_{\mathbf{Z}} a$ and for every $b \notin R$, $R, b \nvdash_{\mathbf{Z}} a$ (such a theory R is said to be relatively maximal). We have proved that in a logic with a classical implication every relatively maximal theory is a maximal theory (see Béziau 95, Béziau 98). Therefore, as the relation $\vdash_{\mathbf{Z}}$ is monotonic and finite and as $\to$ is a classical implication in **HZ**, the above Lindenbaum Lemma holds for **HZ**. □

COROLLARY 3.1. *$T \vdash_{\mathbf{Z}} a$ iff $a \in M$ for every maximal theory M which is an extension of T. (In particular $\vdash_{\mathbf{Z}} a$ iff $a \in M$ for every maximal theory M.)*

LEMMA 3.3. *If M is a maximal theory then we have:*

(i) *$a \in M$ iff $M \vdash_{\mathbf{Z}} a$.*

(ii) *If $a \notin M$, then $\neg a \in M$.*

[2] As usually, the case $n = 0$ means that $\vdash_{\mathbf{Z}} a$

(iii) *If* $\neg a \notin M$*, then* $a \in M$*.*

(iv) $a \vee b \in M$ *iff* $a \in M$ *or* $b \in M$*.*

(v) $a \wedge b \in M$ *iff* $a \in M$ *and* $b \in M$*.*

(vi) *If* $a \to b \in M$ *and* $a \in M$*, then* $b \in M$*.*

(vii) *If* $a \notin M$*, then* $\neg\neg a \notin M$*.*

DEFINITION 3.5. Given a theory T, the theory $\#T = \{b : \neg b \notin T\}$ is called the *center* of T. If $\#T \neq \emptyset$, we say that T *has a center*, in the other case that T *has no center*.

THEOREM 3.2. *Every maximal theory has a center.*

PROOF. Let be M a maximal theory and a such that $M \nvdash_{\mathbf{Z}} a$, then $a \notin M$, then $\neg\neg a \notin M$, therefore, $\neg a \in \#M$. □

DEFINITION 3.6. Given a theory T, a theory C which contains the center of T is called a *companion* of T.

THEOREM 3.3. *If a maximal theory* U *contains the center of a maximal theory* M*, then the center of* U *contains the center of* M*.*

PROOF. Suppose that $\#M \subseteq U$, $a \in \#M$. If $a \notin \#U$, then $\neg a \in U$. As u is maximal, u has a center. Let $b \in \#U$, $b \wedge \neg a \in U$. As $a \in \#M$, $\neg a \notin M$, therefore $b \wedge \neg a \notin M$, thus $\neg\neg(b \wedge \neg a) \notin M$, and we have $\neg(b \wedge \neg a) \in \#M$. As $\#M \subseteq U$, then $\neg(b \wedge \neg a) \in U$. We have $(b \wedge \neg a) \wedge \neg(b \wedge \neg a) \in U$. Using axiom (AZ2), we get $b \wedge \neg b \in U$, but this is absurd, because as $b \in \#U$, $\neg b \notin U$. □

COROLLARY 3.2. *The relation of companionship between maximal theories is transitive.*

THEOREM 3.4. *The relation of companionship between maximal theories is reflexive.*

PROOF. Given a maximal theory M, we have to show that M is a companion of itself, i.e. $\{a : \neg a \notin M\} \subseteq M$.

If $\neg a \notin M$, then $a \in M$ (by the effect of the excluded middle). □

THEOREM 3.5. *The relation of companionship between maximal theories is symmetric.*

PROOF. We have to show that given two maximal theories M and U. If $\{a : \neg a \notin M\} \subseteq U$ then $\{a : \neg a \notin U\} \subseteq M$. Suppose $a \notin M$, then by (6), $\neg\neg a \notin M$, so $\neg a \in U$. □

LEMMA 3.4. *Given a maximal theory T and a formula a: if $a \in M$ for every maximal companion M of T, then $\neg a \notin T$.*

PROOF. If $a \in M$ for every maximal theory which contains the set $B = \{b : \neg b \notin T\}$, therefore by Corollary 3.1:

$$B \vdash_{\mathbf{Z}} a\,.$$

Then there are formulas $b_1, \ldots, b_n$ of B such that:

$$\vdash_{\mathbf{Z}} b_1 \wedge \cdots \wedge b_n \rightarrow a\,.$$

By the rule (RZ), we have:

$$\vdash_{\mathbf{Z}} \neg(b_1 \wedge \cdots \wedge b_n \wedge \neg a).$$

Therefore, as T is maximal, by Corollary 3.1 we have:

$$\neg(b_1 \wedge \cdots \wedge b_n \wedge \neg a) \in T\,.$$

By definition $\neg b_i \notin T$, $(1 \leq i \leq n)$, therefore using (AZ1), it is easy to see that $b_i \in T$.

Now suppose that $\neg a \in T$, then $b_1 \wedge \cdots \wedge b_n \wedge \neg a \in T$.

$$b_1 \wedge \cdots \wedge b_n \wedge \neg a \wedge \neg(b_1 \wedge \cdots \wedge b_n \wedge \neg a) \in T\,.$$

Therefore by (AZ2):

$$b_1 \wedge \cdots \wedge b_n \wedge \neg(b_1 \wedge \cdots \wedge b_n) \in T\ ,.$$

Thus

$$\neg(b_1 \wedge \cdots \wedge b_n) \in T\,.$$

Using (AZ3_n) we get

$$\neg b_1 \vee \cdots \vee \neg b_n \in T\,.$$

Therefore there is a i $(1 \leq i \leq n)$ such that $\neg b_i \in T$, which is absurd. Hence $\neg a \notin T$. □

DEFINITION 3.7. The *canonical structure* is the set of all bivaluations which are characteristic functions of maximal theories of **HZ** (given such a bivaluation α, we write M_α the corresponding theory).

Moreover there is a binary relation (relation of *accessibility*) between bivaluations which is defined by:

$$\alpha \mathsf{R} \beta \quad \text{iff} \quad M_\alpha \text{ is a companion of } M_\beta.$$

THEOREM 3.6. *In the canonical structure, we have:*

$$\alpha(\neg a) = 0 \quad \textit{iff} \quad \textit{for every } \beta \textit{ such that } \alpha \mathsf{R} \beta, \textit{ then } \beta(a) = 1.$$

PROOF. This is a corollary of Lemma 3.4. □

THEOREM 3.7. *The accessibility relation of the canonical structure is an equivalence relation.*

PROOF. We have seen that the companionship relation between maximal theories is reflexive, symmetric and transitive. □

COROLLARY 3.3. *If a is not a consequence of T in the canonical structure, then $T \not\models_{\mathbf{Z}} a$.*

PROOF. This corollary results from the two above theorems and the fact that the notion of semantical consequence defined with structures with an accessibility relation which is an equivalence is the same as the one defined by structures, like **Z**-cosmoses, with a universal relation of accessibility, i.e. no relation of accessibility. □

THEOREM 3.8 (Completeness). *If $T \models_{\mathbf{Z}} a$ then $T \vdash_{\mathbf{Z}} a$.*

PROOF. Suppose that $T \nvdash_{\mathbf{Z}} a$, then by Lemma 3.2 there exists a maximal extension M of T such that $a \notin M$. Let μ be the characteristic function of M, then in the canonical structure, we have $\mu(M) = 1$ and $\mu(a) = 0$. Therefore a is not a consequence of M in the canonical structure. Using Corollary 3.3, we conclude that $T \not\models_{\mathbf{Z}} a$. □

Postscript

This paper was written in December 1997 and presented at the Jaśkowski Memorial Symposium in Toruń, July 1998. The present version is nearly identical to the original one, with some minor improvements. I would like to thank MM. Nasieniewski and Pietruszczak for helping me to improve the paper and editing it. Details about how I got the idea of the logic **Z** and the subsequent evolution of my work which led me to a reformulation of the square of opposition into a three-dimensional object can be found in my paper "Adventures in the paraconsistent jungle" (to appear in the forthcoming Handbook of negation and paraconsistency, Elsevier, 2006). I realized later on that the logic Z presented here is equivalent to the logic **S5**; see my paper: "S5 is a paraconsistent logic and so is first-order classical logic" (Logica Studies 9, 2002). But despite of this fact the interest of the paper remains intact and is in some sense even greater, since the axiomatic system for Z is an axiomatization of **S5** using as only primitive connectives, conjunction, disjunction, implication and the paraconsistent negation which corresponds to the classical negation of necessity of **S5**. The main result of the paper is a completeness proof for this axiomatic system, inspired by, but different from, the standard proofs of completeness in modal logic. Moreover in this paper I used a presentation of possible world semantics with bivaluations instead of possible worlds. I have emphasized in a series of talks and papers the philosophical and logical import of this technique: "Many-valued and Kripke Semantics" in *The age of alternative logics*, J. van Benthem et al. (eds), Springer, 2006, "Possible worlds: a fashionable nonsense?" (with Darko Sarenac), talk presented at UC Berkeley, October 2001, "Kripke structures without possible worlds", talk presented at "Semantics and meaning: workshop with Saul Kripke", Campinas, July, 2005.

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